M303 Further Pure Mathematics

Exam 11 June 2019

Unofficial Solutions - these have NOT been checked by the module team. They will likely contain mistakes and omissions but may be helpful for revision purposes.

(Student version)

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Q 1.

(a) By using the Euclidean Algorithm, we obtain:

$$\begin{aligned} 483 &= 2 \times 217 + 49 \\ 217 &= 4 \times 49 + 21 \\ 49 &= 2 \times 21 + 7 \\ 21 &= 3 \times 7 + 0 \end{aligned}$$

Hence hcf(217,483) is equal to the last non-zero remainder, namely 7.

(b) To satisfy the three congruences we must have

$$x \equiv 3 \pmod 3 \tag{1.1}$$
$$3x + 4 \equiv 6 \pmod 7 \tag{1.2}$$
$$7x \equiv 6 \pmod{11} \tag{1.3}$$

Solving (1.2) for x gives

$$3x \equiv 2 \equiv 9 \pmod 7 \Rightarrow x \equiv 3 \pmod 7$$

This, together with (1.1), gives $x \equiv 3 \pmod{28}$ since $\text{hcf}\,(4, 7) = 1$, by Cor 1.7 (HB p19). Hence $x = 3, 31, 59, \ldots$.

The third congruence can be rearranged as follows

$$\begin{aligned} & 7x \equiv 6 \pmod{11} \\ \Leftrightarrow\ & 7x \equiv 28 \pmod{11} \\ \Leftrightarrow\ & x \equiv 4 \pmod{11} \end{aligned}$$

Therefore we have

$x = 4, 15, 26, 37, 48, \mathbf{59}, \ldots$

Thus, the least positive solution that satisfies all three linear congruences is 59.

(c) The statement is false since the number $87 = 3 \times 29$ has the form $17m + 2$ for $m = 5$ (since $17 \times 5 + 2 = 87$) despite the fact that neither 3 nor 29 is a prime divisor of this same form.

Q 2.

(a)

(i) $|G| = 9 \times 5 = 45$.

'Write down' so no reason needed, but Prop 2.10 HB p 32 applies.

(ii) The order of (3,0) in G is 3.

(iii) (0, 1)

(iv) G is cyclic, it is the direct product of groups of coprime order (9 and 5). (2.11, HB.p32)

(b)

(i) $G = \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ (or $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$),
$H = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.
(The re-ordering is helpful but not required.)
Since G and H are not isomorphic to the same direct product of cyclic groups of prime-power order, they are not isomorphic. For instance, G has an element of order $2^2 = 4$; H has no such element. This is a consequence of (2.13, HB p32).

(ii) No. Neither G nor H is cyclic, but $\mathbb{Z}_{360}$ is cyclic. (See 2.11, HB p32).

(iii) No, because G and H are Abelian (since they are direct products of Abelian groups), but is Dic_{90} is not Abelian (by HB p34).

Q 3.

(a) Using results from HB Page 46, as noted below, we have:

(127/167)	Rule used, HB p46
$= -(167/127)$	Thm 4.2
$= -(40/127)$	2.2(a)
$= -(4/127)(10/127)$	2.2(c)
$= -1 \times (10/127)$	2.2(b)
$= -1 \times (5/127)$	Prop 3.2
$= -(127/5)$	Thm 4.2
$= -(2/5)$	2.2(a)
$= -(-1) = 1$	Thm 4.2

Hence, 127 is a quadratic residue of 167.

(b) From "Solutions of quadratic congruences" (HB, p45), $3x^2 + 4x + 5 \equiv 0 \pmod{17}$ has solutions if, and only if, $b^2 - 4ac$ is congruent modulo 17 to a square, where $a = 3, b = 4$ and $c = 5$.
Now, $b^2 - 4ac = 4^2 - 4(3)(5) = 16 - 60 = -44$.
From HB p46,

$$\begin{aligned}(-44/17) &= (4/17)(-11/17) \quad (\text{ Thm 2.2c})\\ &= 1 \times (6/17) \quad (\text{ Thm 2.2b \& Thm 2.2a})\\ &= (2/17)(3/17) \quad (\text{ Thm 2.2c})\\ &= 1 \times -1 = -1 \quad (\text{ Prop 3.2 \& Prop 4.7})\end{aligned}$$

Hence $3x^2 + 4x + 5 \equiv 0 \pmod{17}$ has no solutions as -44 is not congruent modulo 17 to a square.

(c) By the Rational Root Test (Thm 4.7, HB p52), any rational root of $f(x)$ will have the form $\frac{p}{q} \in \mathbb{Q}$ where $\text{hcf}(p, q) = 1$ and $p \mid 3$ and $q \mid 1$.
Therefore $p \in \{\pm 1, \pm 3\}$ and $q \in \{\pm 1\}$.
Thus possible roots are $1, -1, 3, -3$.
Now
$f(1) = 1 + 12 - 5 + 3 = 11 \neq 0$,
$f(-1) = -1 + 12 + 5 + 3 = 19 \neq 0$,
$f(3) = 27 + 108 - 15 + 3 = 123 \neq 0$,
$f(-3) = -27 + 108 + 15 + 3 = 99 \neq 0$,
Hence $f(x)$ has no rational root and so no linear factor. Therefore, since $f(x)$ has degree 3, $f(x)$ must be irreducible over $\mathbb{Q}$.

Q 4.

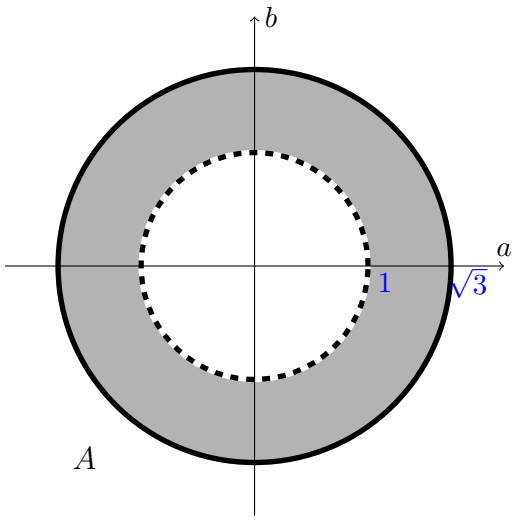
$A = \{(a, b) \in \mathbb{R}^2 : 1 < a^2 + b^2 \leq 3\}$

$B = \{(a, b) \in \mathbb{R}^2 : a > 0\}$

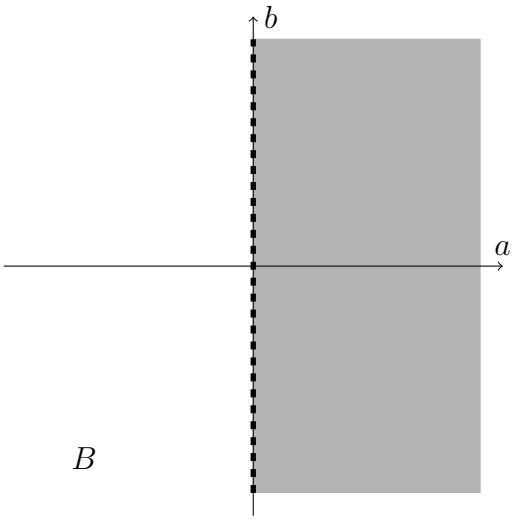
$C = A - B = \{(a, b) \in \mathbb{R}^2 : 1 < a^2 + b^2 \leq 3 \text{ and } a \leq 0\}$

(a) A is the annulus with points on the circle of radius $\sqrt{3}$ centred at the origin included and points on the circle of radius 1 centred at the origin excluded.

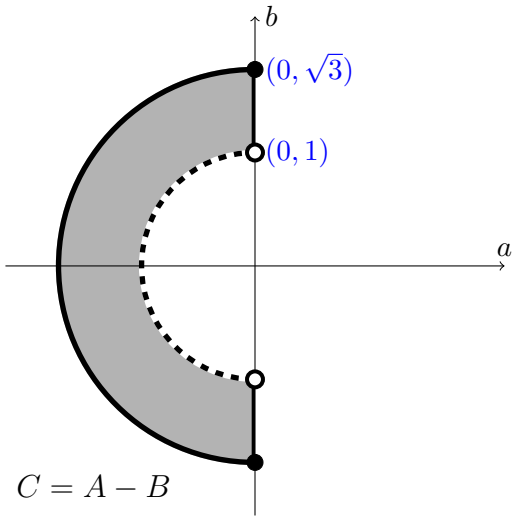
Question asks for a sketch **or** a description, so obviously only one needed, but both given here for completeness.



B is the half-plane excluding points on the b-axis.



C is the semi-annulus with points on the (left vertical) semi-circle of radius $\sqrt{3}$ centred at the origin included and points on the (left vertical) semi-circle of radius 1 centred at the origin excluded.



(b) With metric space $(\mathbb{R}^2, d^{(2)})$ and $C \subset \mathbb{R}^2$, we show that $(-1, 0)$ is a closure point of C for the Euclidean metric $d^{(2)}$.
Consider the sequence (c_n) given by $c_n = (-1 - 1/2n, 0)$ for $n \in \mathbb{N}$.
Since $(1/n)$ is a basic null sequence (Thm 1.13, HB p63) and $-1/2$ is a constant multiple, then $(-1/2n)$ is a null sequence by Thm 1.14 (HB p63). Thus, by Defn 3.2 (HB, p73) the sequence (c_n) is $d^{(2)}$-convergent to the point $(-1, 0) \in \mathbb{R}^2$, with all the terms of (c_n) lying in C.
Then by Cor 3.11 (HB p84), the point $(-1, 0)$ is a closure point of C for the Euclidean metric $d^{(2)}$.

(c)

$$\begin{aligned}
\mathrm{Cl}_{(\mathbb{R}^2, d^{(2)})}(C) &= \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 3 \text{ and } a \leq 0\} \\
\mathrm{Int}_{(\mathbb{R}^2, d^{(2)})}(C) &= \{(a, b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 3 \text{ and } a < 0\} \\
\mathrm{Bd}_{(\mathbb{R}^2, d^{(2)})}(C) &= \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1 \text{ and } a < 0\} \cup \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 3 \text{ and } a < 0\} \\
&\quad \cup \{(0, b) \in \mathbb{R}^2 : 1 \leq |b| \leq \sqrt{3}\}
\end{aligned}$$

(d) The closure of C is $d^{(2)}$- closed (Basic properties of closure, HB p84).
Also, since the closure of C is a subset of the $d^{(2)}$-ball of radius $\sqrt{3}$ centred at the origin, any two points in the closure of C can be at most a $d^{(2)}$distance of $2\sqrt{3}$ apart.
So with $M = 2\sqrt{3} > 0$, $\mathrm{Cl}_{(\mathbb{R}^2, d^{(2)})}(C)$ is a $d^{(2)}$-bounded set by Defn 2.9 (HB, p129).
It then follows, by the Heine-Borel Theorem (Thm 5.14, HB p134), that $\mathrm{Cl}_{(\mathbb{R}^2, d^{(2)})}(C)$ is $d^{(2)}$-compact. That is, the closure of C for the Euclidean metric on the plane is compact, as required.

Q 5.

(a) We are given $\left[\mathbb{Q}\left(\sqrt{8}\right):\mathbb{Q}\right] = \left[\mathbb{Q}\left(\sqrt{7}\right):\mathbb{Q}\right] = 2$ and $\sqrt{7} \notin \mathbb{Q}\left(\sqrt{8}\right)$, $\sqrt{8} \notin \mathbb{Q}\left(\sqrt{7}\right)$.
Let $\alpha = \sqrt{8} - \sqrt{7}$. Then:

$$\begin{aligned}\alpha^3 &= \left(\sqrt{8} - \sqrt{7}\right)^3 \\ &= (\sqrt{8})^3 - 3(\sqrt{8})^2\left(\sqrt{7}\right) + 3\sqrt{8}\left(\sqrt{7}\right)^2 - \left(\sqrt{7}\right)^3 \\ &= 8\sqrt{8} - 24\sqrt{7} + 21\sqrt{8} - 7\sqrt{7} \\ &= 29\sqrt{8} - 31\sqrt{7}.\end{aligned}$$

Therefore $a = 29$ and $b = -31$.

(b) Now $\alpha^3 - 29\alpha = -2\sqrt{7}$ and from the field axioms $-\frac{1}{2}\left(\alpha^3 - 29\alpha\right) \in \mathbb{Q}(\alpha)$.
Thus $\sqrt{7} \in \mathbb{Q}(\alpha)$.
Using this, and the field axioms, $\sqrt{8} = \alpha + \sqrt{7} \in \mathbb{Q}(\alpha)$.

(c) Since, by part (b), $\sqrt{7}, \sqrt{8} \in \mathbb{Q}(\alpha)$ it follows that
$\mathbb{Q}(\sqrt{7}, \sqrt{8}) \subseteq \mathbb{Q}(\alpha)$.
For the reverse inclusion, we note that $\alpha = \sqrt{8} - \sqrt{7} \in \mathbb{Q}(\sqrt{7}, \sqrt{8})$ from which it follows that $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{7}, \sqrt{8})$.
Hence $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{7}, \sqrt{8})$.

(d) Applying the KLM Theorem (HB p105), with $\mathbb{Q} \subset \mathbb{Q}(\sqrt{7}) \subset \mathbb{Q}(\sqrt{7}, \sqrt{8})$, we have

$$[\mathbb{Q}(\sqrt{7}, \sqrt{8}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{7}, \sqrt{8}) : \mathbb{Q}(\sqrt{7})][\mathbb{Q}(\sqrt{7}) : \mathbb{Q}]$$

The rôles of $\sqrt{7}$ and $\sqrt{8}$ may be interchanged here.

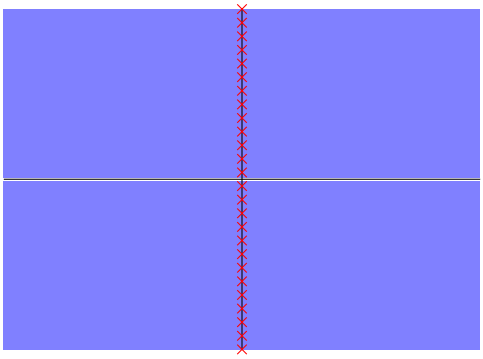
We are given that $\sqrt{8} \notin \mathbb{Q}(\sqrt{7})$ therefore the minimal polynomial for $\sqrt{8}$ over $\mathbb{Q}(\sqrt{7})$ is $x^2 - 8$. Hence
$[\mathbb{Q}(\sqrt{7}, \sqrt{8}) : \mathbb{Q}(\sqrt{7})] = 2$, by Thm 2.21(c), HB p105.
Further, we are given that $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}] = 2$ and so $[\mathbb{Q}(\sqrt{7}, \sqrt{8}) : \mathbb{Q}] = 4$
Hence by part (c) above, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.

(e) $\alpha^2 = \left(\sqrt{8} - \sqrt{7}\right)^2 = 8 - 2\sqrt{7}\sqrt{8} + 7 = 15 - 2\sqrt{7}\sqrt{8}$
and so $\alpha^2 - 15 = -2\sqrt{7}\sqrt{8}$ giving $(\alpha^2 - 15)^2 = \left(-2\sqrt{7}\sqrt{8}\right)^2 = 224$.
Hence $\alpha^4 - 30\alpha^2 + 225 = 224$, that is $\alpha^4 - 30\alpha^2 + 1 = 0$.
Since, by part(d), the minimal polynomial for α over $\mathbb{Q}$ must have degree 4, it follows that $x^4 - 30x^2 + 1$ is the required minimal polynomial. (It is clearly monic.)

Q 6.

(a) $d\big((-1,-1),(1,1)\big) = |-1-1|+2|-1-1| = 2+2\times 2 = 6$

(b) b) Here is a sketch of X. It consists of the whole plane, excluding the red dotted line.



Sketch not essential, but it does help.

A d-disconnection of X is $\Big\{\big\{(a,b)\in\mathbb{R}^2 : a<0\big\}, \big\{(a,b)\in\mathbb{R}^2 : a>0\big\}\Big\}$.

(c)

(i) **NB There was a correction issued to question 6(c)(i).**
Instead of saying:

"Show that for $(a,b)\in X, d\left((1/n,0),(a,b)\right)\geq 2\,|b|$."

The question should have said:

"Show that for $(a,b)\in X, d\left((1/n,0),(a,b)\right)\geq \frac{|a|}{2}$ for $n>\frac{2}{|a|}$."

Using the given metric,

$$\begin{aligned} d\Big((1/n,0),(a,b)\Big) &= |1/n-a|+2\,|0-b| \\ &= |1/n-a|+2\,|b| \\ &\geq |1/n-a| \\ &\geq \big||1/n|-|a|\big| \text{ by reverse triangle inequality} \\ &\geq |a|-\frac{|a|}{2} \text{ for } n>\frac{2}{|a|} \\ &= \frac{|a|}{2} \end{aligned}$$

Thus, $d\Big((1/n,0),(a,b)\Big)\geq \frac{|a|}{2}$ for $n>\frac{2}{|a|}$.

(ii) We give a proof by contradiction.
Take any $(a,b)\in X$ and suppose that $(\frac{1}{n},0)$ is d-convergent to (a,b).
Then, using part (c)(i), we note that, since $(a,b)\in X, d\big((1/n,0),(a,b)\big)\geq |a|/2$.
That is, the sequence $\Big(d\big((1/n,0),(a,b)\big)\Big)$ cannot be a real null sequence.
It follows that $(a_n)=(1/n,0)$ cannot be d-convergent to any point $(a,b)\in X$.

(iii) We will show that the sequence (a_n) where $a_n = (1/n, 0)$ is d-Cauchy, but not d-convergent. The latter follows from the previous part; thus we need only prove that (a_n) is d-Cauchy. Take any $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $N > 2/\varepsilon$. Then

$$\begin{aligned} d(a_n, a_m) &= |1/n - 1/m| + 2|0 - 0| \\ &= |1/n - 1/m| \\ &\leq 1/n + 1/m < \frac{2}{N} < \varepsilon, \text{ provided } m, n > N. \end{aligned}$$

It follows that (a_n) is d-Cauchy.
Finally, since (a_n) is a d-Cauchy sequence that is not d-convergent we conclude that X is not d-complete, by Defn 2.1 (HB p136).

(iv) By Thm 2.3 (HB p136) since X is not complete, it is not compact. Then, by Thm 5.10 (HB p133) since it is not compact, it is not sequentially compact.

Q 7.

(a)

(i) $r^{25}sr^{25}s = r^{25}ssr^{-25} = r^{25}s^2r^{-25} = r^{25}er^{-25} = r^{25}r^{-25} = e.$

(ii) We already know that r^{25} commutes with every element of D_{50} having the form r^i where $i = 0, 1, \ldots, 49$.
Furthermore, we observe that $r^{25}s = sr^{-25} = sr^{25}$ since $r^{25} = (r^{25})^{-1} = r^{-25}$.

Hence, we can write
$r^{25}(r^is^j) = (r^{25}r^i)s^j = (r^ir^{25})s^j = r^i(r^{25}s^j) = r^i(s^jr^{25}) = (r^is^j)r^{25}$, where $j = 0, 1$.

Therefore, r^{25} commutes with each element of D_{50} implying that $r^{25} \in Z(D_{50})$.

(b)

(i) The order of the Sylow 2-subgroups is 4 and the order of the Sylow 5-subgroups is 25. 'Write down', so "4, 25" gets the mark.

(ii) If there are m_2 Sylow 2-subgroups, then we know that

$$m_2 \equiv 1 \pmod 2 \text{ and } m_2 \Big| 25.$$

Hence the possible numbers of Sylow 2-subgroups are 1,5 and 25.

(iii) If there are m_5 Sylow 5-subgroups, then we know that

$$m_5 \equiv 1 \pmod 5 \text{ and } m_5 \Big| 4.$$

The only solution is $m_5 = 1$.
Considering Lemma 1.9, (HB, p39) and the fact that G has only one subgroup of order 25, then we deduce that this subgroup is normal.

(c)

(i) If there are m_p Sylow p-subgroups, then we know that $m_p \equiv 1 \pmod p$, which entails that m_p has the form $kp + 1$, where k is a non-negative integer.
Additionally, $m_p \Big| q^2$ and the only positive divisors of q^2 are $1, q$ and q^2.
But here we are assuming that $m_p \neq 1$ which leaves only two possibilities.
If $m_p = q$ then $kp + 1 = q$, which is a contradiction as $p > q$ and $k \geq 1$.
Therefore, the only possibility for m_p is $m_p = q^2$.
In other words, $kp + 1 = q^2$ or $kp = q^2 - 1$ from which it follows that p divides $q^2 - 1$.

(ii) If p divides $q^2 - 1$ then we can write $kp = (q-1)(q+1)$, leading us to the conclusion that p must divide at least one of $q + 1$ or $q - 1$. However p can not divide $q - 1$ since $p > q$.
By the same reasoning p can not divide $q + 1$ since $q + 1$ is not equal or greater than p (the difference of two odd prime numbers must be at least 2).
Hence, we end up in a contradiction and our initial assumption (that p divides $q^2 - 1$) is wrong. Therefore, m_p must be equal to 1 and G has a unique Sylow p-subgroup.

(iii) By Lemma 1.9, (HB p39) and the fact that G has only one subgroup of order p^2, we deduce that this subgroup is normal.
Further, as $1 < p^2 < p^2q^2 = |G|$ this is a non-trivial subgroup of G.
Thus this Sylow p-subgroup is a non-trivial proper normal subgroup and by Defn 4.4, (HB p38) we conclude that G cannot be simple.

Q 8.

We are given $R = \mathbb{Z}\left[\sqrt{-6}\right] = \left\{a + b\sqrt{-6} : a, b \in \mathbb{Z}\right\}$ and $N(a + b\sqrt{-6}) = a^2 + 6b^2$ is a multiplicative norm on R.

(a) For $r = a + b\sqrt{-6} \in R$. If $N(r) = 1$, then $a^2 + 6b^2 = 1$.
Thus $b^2 = 0$, else $N(r) \geq 6$, so $a^2 = 1$ giving $a = \pm 1$, hence $r = \pm 1$ as required.
Suppose now $N(r) = 2$ or $N(r) = 3$. Then as above, we must have $b = 0$, but $a^2 = 2$ and $a^2 = 3$ have no integer solutions.
Hence there are no elements r with $N(r) = 2$ or $N(r) = 3$.

(b) Suppose that $2 = st$ for some $s, t \in R$. Then

$$N(st) = N(s)N(t) = N(2) = 4$$

The integer factors of 4 are 1, 2, 4. But by (a) we cannot have $N(s) = N(t) = 2$. Hence one of $N(t)$ or $N(s)$ equals 1, so again by (a), s or t equals ± 1 and is a unit, and so 2 is irreducible in R, by definition.
A similar argument applies for $3 \in R$.
We now consider the case of $\sqrt{-6}$.
Suppose that $\sqrt{-6} = st$, for some $s, t \in R$. Then

$$N(st) = N(s)N(t) = N(\sqrt{-6}) = 6$$

and the integer factors of 6 are 1, 2, 3, 6. However from (a) we have that no elements of R have norm equal to 2 or 3.
Hence again, one of s or t is a unit, and so $\sqrt{-6}$ is irreducible in R, as required.

(c) To show R is not a UFD, we will find an element that is irreducible but not prime (Cor 4.3, HB p56).
Considering $\sqrt{-6} \in R$. Now $(\sqrt{-6})(-\sqrt{-6}) = 2 \times 3 = 6$, hence $\sqrt{-6}$ divides 6, but $\sqrt{-6} \nmid 2$, and $\sqrt{-6} \nmid 3$.

Hence $\sqrt{-6}$ is irreducible but not prime, and so R is not a UFD.

(d) Suppose that $7 = uv$, where u and v are non-units in R.
Taking norms:

$$N(uv) = N(u)N(v) = N(7) = 49,$$

and the only proper integer factorization of 49 is 7×7. Hence, since, u, v are not units, we must have $N(u) = N(v) = 7$.

(e) Take any $u = a + b\sqrt{-6} \in R$, with $N(u) = 7$, i.e. $a^2 + 6b^2 = 7$.
Then $b^2 = 0$ or $b^2 = 1$,since otherwise else $N(u) \geq 24$. However, if $b^2 = 0$, then $a^2 = 7$, which has no integer solutions.
So $b^2 = 1$, and we have $a^2 + 6 = 7$, giving solutions of $a = \pm 1, b = \pm 1$.

Hence the solution set is $\left\{1 + \sqrt{-6},\ 1 - \sqrt{-6},\ -1 + \sqrt{-6},\ -1 - \sqrt{-6}\right\}$.

(f) By (d), for any $u, v \in R$ with $uv = 7$, we must have $N(u) = N(v) = 7$, which has four possible factors by (e).
Then we have
$(1 + \sqrt{-6})(1 - \sqrt{-6}) = (-1 + \sqrt{-6})(-1 - \sqrt{-6}) = 7$.
But $(1 + \sqrt{-6}), (-1 - \sqrt{-6})$, and $(1 - \sqrt{-6}), (-1 + \sqrt{-6})$ pair off as associates.
Hence the factorization of 7 in R is unique.

(g) Now $I = \langle 2, 1 + \sqrt{-6} \rangle$ by definition is given by

$I = \{2r + (1 + \sqrt{-6})s : r, s \in R\}$

But both -3 and $1 - \sqrt{-6}$ are in R, so

$2 \times (-3) + (1 + \sqrt{-6}) \times (1 - \sqrt{-6}) = -6 + 7 = 1 \in R.$

Then by Properties of Ideals (HB, p94), I is not a proper ideal, that is, $I = R$.

Q 9.

(a) Here:
$A = \{f \in C[0,1] : |f(x) - f(y)| \leq \exp(|x-y|)|x-y| \text{ for each } x, y \in [0,1]\}$
and
$B = \{f \in C[0,1] : |f(x) - 2019| \leq 1 \text{ for } x \in [0,1]\}$.

(i) To show that A is $d_{\max}$-closed, let (f_n) be a sequence in A such that $f_n \xrightarrow{d_{\max}} f \in C[0,1]$. We must show that $f \in A$. That is, we must show that

$$|f(x) - f(y)| \leq \exp(|x-y|)|x-y| \text{ for each } x, y \in [0,1].$$

Now, for each x, $y \in [0, 1]$

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\quad \text{by the Triangle Inequality.} \end{aligned}$$

Since $x, y \in [0,1]$, $|f(x) - f_n(x)| \leq d_{\max}(f_n, f)$ and $|f(y) - f_n(y)| \leq d_{\max}(f_n, f)$.

Also, since each $f_n \in A$,

$|f_n(x) - f_n(y)| \leq \exp(|x-y|)|x-y|$ for each $x, y \in [0,1]$.

Hence,

$|f(x) - f(y)| \leq 2d_{\max}(f_n, f) + \exp(|x-y|)|x-y|$ for each $x, y \in [0,1]$.

But $d_{\max}(f_n, f) \to 0$ as $n \to \infty$, so

$$|f(x) - f(y)| \leq \exp(|x-y|)|x-y| \text{ for each } x, y \in [0,1].$$

That is, $f \in A$ and therefore A is $d_{\max}$-closed, as required.

(ii) From (i), A is $d_{\max}$-closed.
Also, $B = \{f \in C[0,1] : |f(x) - 2019| \leq 1 \text{ for } x \in [0,1]\}$ is $d_{\max}$-closed by the examples in Appendix 5 (HB p89).
Then $A \cap B$ is the intersection of $d_{\max}$-closed sets and is therefore also $d_{\max}$-closed, as required.

(iii) We will use the Arzelà-Ascoli Thm (HB, p131) for $A \cap B \subseteq C[0,1]$.
Now $A \cap B$ is $d_{\max}$-closed, by part (ii) above.
Also, from the definition of B, $|f(x) - 2019| \leq 1 \Rightarrow 2018 \leq f(x) \leq 2020$.
Hence B and so $A \cap B$ is pointwise bounded.
Finally, we are given that A is equicontinuous and since $A \cap B \subseteq A$, it follows that $A \cap B$ is equicontinuous also (Example 2, HB p131).
Thus $A \cap B$ satisfies the three conditions of the Arzelà-Ascoli Theorem, and so is $d_{\max}$-sequentially compact, as required.

(b) This distance function can be thought of as:

$$d(a,b) = \begin{cases} 0 & \text{if both zero} \\ 1 + \max\{a,b\} & \text{if one is zero} \\ |a-b| & \text{if neither is zero} \end{cases}$$

for $a, b \in [0, \infty)$.

(i) $d(0,1) = 1 + \max\{0,1\} = 1 + 1 = 2$.

(ii) We must show d satisfies M1, M2, M3.

M1: Non-negativity
Each of 0, $1 + \max\{a,b\}$, $|a-b|$ is not less than 0 for $a, b \in [0, \infty)$.
If $a = b$, then either:
$a = b = 0$ and we have, from the definition, $d(a,b) = d(0,0) = 0$,
or
$a = b \neq 0$ and again, from the definition, $d(a,b) = |a-b| = |a-a| = 0$.
Finally, if $d(a,b) = 0$ then either $a = b = 0$ or $|a-b| = 0$. In either case, $a = b$.
Hence (M1) holds.

M2: Symmetry
If both a, b are zero then $d(0,0) = 0 = d(0,0)$ is trivially true.
If one of a, b is zero, suppose WLOG that $b = 0$.
Then $d(a,b) = d(a,0) = 1 + a = d(0,a) = d(b,a)$.
If neither a nor b is zero, then $d(a,b) = |a-b| = |b-a| = d(b,a)$.
Hence (M2) holds.

M3: Triangle inequality

For $a, b, c \in [0, \infty)$
If $a = b = c = 0$ then trivially $d(a,c) = 0 = 0 + 0 = d(a,b) + d(b,c)$.

If none of a, b, c equals zero, then $d(a,c) = |a-c| \leq |a-b| + |b-c| = d(a,b) + d(b,c)$ by the triangle equality for $d^{(1)}$ (that is the ordinary modulus).

If $a = 0$ and $b, c \neq 0$ then

$$\begin{aligned} d(a,c) &= 1 + c \\ &= 1 + b - b + c \\ &= 1 + b + (c - b) \\ &\leq 1 + b + |c - b| \\ &= 1 + b + |b - c| = d(a,b) + d(b,c) \end{aligned}$$

A similar proof covers the case $c = 0$ and $a, b \neq 0$.

If $b = 0$ and $a, c \neq 0$ then

$$\begin{aligned} d(a,c) &= |a - c| \\ &= |1 + a - 1 - c| \\ &= |1 + a - (1 + c)| \\ &\leq |1 + a| + |1 + c| \\ &= (1 + a) + (1 + c) \\ &\leq 1 + \max\{a,b\} + 1 + \max\{b,c\} = d(a,b) + d(b,c). \end{aligned}$$

If and $a = b = 0$ and $c \neq 0$ then

$$\begin{aligned} d(a, c) &= 1 + \max\{a, c\} \\ &= 0 + 1 + c = d(a, b) + d(b, c) \end{aligned}$$

A similar proof covers the case where $b = c = 0$ and $a \neq 0$.

If and $a, c = 0$ and $b \neq 0$ then

$$d(a, c) = 0 \leq d(a, b) + d(b, c) \text{ by virtue of M1.}$$

Hence (M3) holds.
Therefore d is a metric on $[0, \infty)$.